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The Oscillatory Nature of the Equation $y''' + qy' + ry = 0$

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1. INTRODUCTION

In Oscillation Theory one is interested in knowing if the solutions of a given differential equation are oscillatory in the interval $(0, \infty)$. A solution is said to be oscillatory in $(0, \infty)$ if and only if it has an infinite numbers of zeros there, but it is nonoscillatory otherwise. The equation

$$y''' + qy' + ry = 0, \tag{H}$$

where q and r are real-valued is said to be oscillatory if and only if it has at least one nontrivial oscillatory solution. It is nonoscillatory if all its solutions are nonoscillatory. Another question of interest is, "What happens to the nonoscillatory solutions, if there are any, of an oscillatory equation as $t \to \infty$?"

In this paper we will answer some of the above questions with respect to Eq. (H) on the interval $[a, \infty)$, where $a \geqslant 0$. Equation (H) will be shown to be oscillatory in two cases. The first will be where the coefficient q has no restrictions on it with respect to sign, but where r must be nonzero on $[a, \infty)$. In the second case, there is no restriction on r, but q must be positive on $[a, \infty)$ and eventually monotonic. In existing theorems the coefficients do not possess this freedom with respect to sign. See Barrett [1], Hanan [2], Lazer [4], and Swanson [6].

The results will be obtained by making use of some corollaries concerning asymptotic solutions of Eq. (H) previously developed by the author [5].

It should be mentioned that, because of the nature of the asymptotic solutions we will be able to determine what happens to the solutions and their first and second derivatives as $t \to \infty$.

Also, the asymptotic solutions will only provide information about the actual solutions on an interval $[t_0, \infty)$ where $t_0 \geqslant a$. The following well-known lemma is needed.

LEMMA 1. *If r and q are continuous on $[a, \infty)$ and if y, a solution of Eq.* (H), *is oscillatory or nonoscillatory on $[t_0, \infty)$ where $t_0 \geq a$, then y is oscillatory or nonoscillatory on $[a, \infty)$.*

Proof. It is obvious that if y is oscillatory on $[t_0, \infty)$, then it is oscillatory on $[a, \infty)$.

Now suppose that y is nonoscillatory on $[t_0, \infty)$ for some $t_0 > a$ and oscillatory on $[a, t_0)$. Then there is a sequence of distinct numbers $\{t_i\}_{i=1}^{\infty}$ in $[a, t_0)$ such that $y(t_i) = 0$ for each i. Since $[a, t_0]$ is closed and bounded, the sequence has a limit point t' in $[a, t_0]$. Because each of y, y', and y'' are continuous on $[a, \infty)$, we have

$$y(t') = y'(t') = y''(t') = 0. \tag{1}$$

Now since r and q are continuous on $[a, \infty)$ we know from the uniqueness theorems of ordinary differential equations that the only solution of Eq. (H) which satisfies (1) is the trivial solution $y(t) = 0$ for all t in $[a, \infty)$. But this is a contradiction since y is nonoscillatory on $[t_0, \infty)$.

Therefore y is nonoscillatory on $[a, \infty)$.

2. r NONZERO AND DOMINANT

Here we will consider the case where $r \neq 0$ and $q/r^{1/3}$ is small on $[a,\infty)$.

From here on we will make use of the symbol $L(a, \infty)$ which refers to the set of all complex-valued functions which are Lebesque integrable.

The following theorem and lemma are needed.

THEOREM 2. (Hinton [3]). *If $r > 0$ on $[a, \infty)$ and $r''/r^{1+1/n}$, for $n = 1, 2,...$ is in $L(a, \infty)$, then*

(i) *$r^{1/n}$ is not in $L(a, \infty)$,*

(ii) *$(r'/r^{1+1/n})'$ is in $L(a, \infty)$, and*

(iii) *$(r'/r^{1+1/2n})^2$ is in $L(a, \infty)$.*

LEMMA 3. *If $r''/r^{4/3}$ is in $L(a, \infty)$ and α is any real number such that $\alpha r^{1/3} > 0$ on $[a, \infty)$ and $\beta = \pm 1/3$, then*

$$r^{\beta} \exp\left(-\alpha \int_a^t r^{1/3}\, d\tau\right) \to 0$$

as $t \to \infty$.

Proof. Assume that $r > 0$ and therefore so is α. First we would like to show that $r'/r^{4/3} \to 0$ as $t \to \infty$. By Theorem 2(ii) we see that $r'/r^{4/3} \to c$, a constant, as $t \to \infty$. Hence $(r'/r^{4/3})^2 \to c^2$ as $t \to \infty$. Assume that $c \neq 0$. Then given $\epsilon = (1/2)\, c^2$ there is a number $a_0 \geqslant a$ such that for $t \geqslant a_0$,

$$(1/2)\, c^2 < (r'/r^{4/3})^2 < (3/2)\, c^2,$$
$$(1/2)\, c^2 r^{1/3} < (r'/r^{7/6})^2 < (3/2)\, c^2 r^{1/3}.$$

But by Theorem 2 (i) and (iii) we see that this cannot hold. Therefore $c = 0$.

Now since $r'/r^{4/3} \to 0$ as $t \to \infty$, there is a number $a_1 \geqslant a$ such that for $t \geqslant a_1$,

$$\beta(\alpha r'/\alpha r) < (\alpha/2)\, r^{1/3},$$

$$\log(\alpha r(t)/\alpha r(a_1))^\beta < (\alpha/2) \int_{a_1}^{t} r^{1/3}\, d\tau,$$

$$0 < (\alpha r(t)/\alpha r(a_1))^\beta < \exp\left((\alpha/2) \int_{a_1}^{t} r^{1/3}\, d\tau\right),$$

and thus

$$0 < (\alpha r(t)/\alpha r(a_1))^\beta \exp\left(-\alpha \int_{a_1}^{t} r^{1/3}\, d\tau\right) < \exp\left(-(\alpha/2) \int_{a_1}^{t} r^{1/3}\, d\tau\right).$$

Applying Theorem 2 (i) to the last inequality above gives

$$(\alpha r(t)/\alpha r(a_1))^\beta \exp\left(-\alpha \int_{a_1}^{t} r^{1/3}\, d\tau\right) \to 0$$

as $t \to \infty$. Finally, multiplication by the proper constants yields the desired results.

For $r < 0$, use the fact that $(-r) > 0$ in Theorem 2 and follow a similar argument.

Now for the main theorem involving the asymptotic solutions of Eq. (H).

THEOREM 4. *If $r \neq 0$ on $[a, \infty)$, and $r''/r^{4/3}$ and $q/r^{1/3}$ are in $L(a, \infty)$, then there are three linearly independent solutions of Eq.* (H) *such that, one solution y_1 is nonoscillatory on $[a, \infty)$ and the other two, y_2 and y_3, are oscillatory on $[a, \infty)$. Furthermore,*

(i) *for $r > 0$ and $j = 0, 1, 2$, $y_1^{(j)} \to 0$ as $t \to \infty$ and $y_2^{(j)}$, $y_3^{(j)}$ are unbounded on $[a, \infty)$, and*

(ii) *for $r < 0$ and $j = 0, 1, 2$, $y_1^{(j)} \to \pm\infty$ as $t \to \infty$ and $y_2^{(j)}$, $y_3^{(j)}$ converge to zero as $t \to \infty$.*

Proof. The hypotheses satisfy Corollary 5 of [5]. By computing the matrices in the results of this corollary one obtains three linearly independent solutions z_1, z_2, and z_3 of Eq. (H) and a number $t_0 \geqslant a$ such that, for $t \geqslant t_0$ and $k = 1, 2, 3$,

$$
\begin{aligned}
z_k(t) &= r^{-1/3} \exp\left(\mu_k \int_{t_0}^{t} r^{1/3}\, d\tau\right)(1 + o(1)),\\
z_k'(t) &= \mu_k \exp\left(\mu_k \int_{t_0}^{t} r^{1/3}\, d\tau\right)(1 + o(1)),\\
z_k''(t) &= \mu_k^2 r^{1/3} \exp\left(\mu_k \int_{t_0}^{t} r^{1/3}\, d\tau\right)(1 + o(1)),
\end{aligned}
\tag{2}
$$

where $\mu_1 = 1$, $\mu_2 = 1/2 + \sqrt{3}\, i/2$, $\mu_3 = 1/2 - \sqrt{3}\, i/2$ and $(1 + o(1)) \to 1$ as $t \to \infty$.

Now set $y_1 = z_1$. Since $r \neq 0$, y_1 is nonoscillatory on $[t_0, \infty)$ and hence on $[a, \infty)$ by Lemma 1.

For the oscillatory solutions, set

$$z_k = \theta_k \exp(i\, \psi_k) \tag{3}$$

for $k = 2, 3$ and where θ_k and ψ_k are real-valued functions on $[a, \infty)$. Let y_2 be the real part of z_2 and y_3 the imaginary part of z_3, i.e.,

$$
\begin{aligned}
y_2 &= \theta_2 \cos \psi_2\,,\\
y_3 &= \theta_3 \sin \psi_3\,.
\end{aligned}
$$

If we can show that ψ_2 and ψ_3 go to $\pm\infty$ as $t \to \infty$, then the solutions y_2 and y_3 will be oscillatory.

Using (2) and (3) we obtain for $k = 2, 3$

$$\frac{\theta_k \exp(i\psi_k)}{r^{-1/3} \exp(\mu_k \int_{t_0}^{t} r^{1/3}\, d\tau)} \to 1$$

as $t \to \infty$. Considering the modulus gives for $k = 2, 3$

$$\theta_k / r^{-1/3} \exp\left((1/2) \int_{t_0}^{t} r^{1/3}\, d\tau\right) \to 1$$

as $t \to \infty$, and for the argument

$$
\begin{aligned}
&\psi^2 - (\sqrt{3}/2) \int_{t_0}^{t} r^{1/3}\, d\tau + 2k\pi \to 0,\\
&\psi_3 + (\sqrt{3}/2) \int_{t_0}^{t} r^{1/3}\, d\tau + 2k\pi \to 0
\end{aligned}
\tag{4}
$$

as $t \to \infty$, where k is some integer.

From Theorem 2 (i) one observes that

$$\int_{t_0}^{t} r^{1/3}\, d\tau \to \pm\infty$$

as $t \to \infty$ depending on whether $r > 0$ or $r < 0$. Using this and (4) we see that ψ_2 and ψ_3 diverge to $\pm\infty$ as $t \to \infty$. Therefore, y_2 and y_3 are oscillatory on $[t_0, \infty)$ and hence on $[a, \infty)$.

It is clear that y_1, y_2, and y_3 as defined are linearly independent solutions of Eq. (H).

To show that (i) and (ii) hold, one uses the fact that

$$| y_k^{(j)} | = | z_k^{(j)} |$$

for $j = 0, 1, 2$ and $k = 1, 2, 3$, in conjunction with (2) and Lemma 3.

An example of an equation covered by this theorem but no other known theorem is

$$y''' + (1/t) \sin t y' + t^3 y = 0.$$

This equation is oscillatory and its solutions satisfy part (i). Notice that $q(t) = (1/t) \sin t$ constantly changes sign.

3. q Positive, Monotonic and Dominant

Here we will consider the case where $q > 0$ and eventually monotonic, and r/q is small on $[a, \infty)$. The restrictions $q > 0$ and monotonic are necessary in order to use a previously developed corollary by the author [5].

THEOREM 5. *If $q > 0$ on $[a, \infty)$, and monotonic on $[a_1, \infty)$ for some $a_1 \geqslant a$, and $q''/q^{3/2}$ and r/q are in $L(a, \infty)$ then there are three linearly independent solutions of Eq.* (H) *such that one solution y_1 is nonoscillatory on $[a, \infty)$ and the other two solutions y_2 and y_3 are oscillatory on $[a, \infty)$. Further,*

(i) $y_1^{(j)}/q^{j/2} \to 1$ *as* $t \to \infty$ *for* $j = 0, 1, 2$,

(ii) y_2, y_2', y_3, and y_3' *are bounded if and only if* q^{-1} *is bounded, and*

(iii) y_2'' *and* y_3'' *are bounded if and only if* q *is bounded.*

Proof. The hypotheses satisfy Corollary 9 of [5]. In particular, q being monotonic satisfies the requirement that $\pm q'/q$ satisfies Condition I. By computing the matrices in the results of this corollary one obtains three linearly independent solutions z_1, z_2, and z_3 of Eq. (H) and a number $t_0 \geqslant a$ such that, for $t \geqslant t_0$,

$$z_1^{(j)}(t)/q^{j/2}(t) \to 1 \tag{5}$$

as $t \to \infty$, and, for $k = 2, 3$ and $j = 0, 1, 2$,

$$z_k^{(j)} = ((-1)^k i)^j \, q^{j/2-3/4} \exp((-1)^k \, i\delta)(1 + o(1)) \tag{6}$$

with

$$\delta(t) = \int_{t_0}^{t} q^{1/2}(1 + (1/16)(q'/q^{3/2})^2)^{1/2} \, d\tau.$$

As in the previous theorem, set $y_1 = z_1$; then y_1 is nonoscillatory on $[t_0, \infty)$ and hence on $[a, \infty)$ by Lemma 1.

For the oscillatory solutions, set

$$z_k = \theta_k \exp(i\psi_k(t)) \tag{7}$$

for $k = 2, 3$ and where θ_k and ψ_k are real-valued functions on $[a, \infty)$. Let y_2 and y_3 be the real and imaginary parts of z_2 and z_3, respectively, i.e.,

$$y_2 = \theta_2 \cos(\psi_2),$$
$$y_3 = \theta_3 \sin(\psi_3).$$

Now show that ψ_2 and ψ_3 diverge to $\pm\infty$, respectively, as $t \to \infty$. Using (6) and (7), we obtain for $k = 2, 3$

$$\frac{\theta_k}{q^{-3/4}} \to 1 \qquad \text{and} \qquad \psi_k + (-1)^{k-1}\,\delta + 2k\pi \to 0$$

as $t \to \infty$ and where k is some integer. Substituting q for r in Theorem 2 tells us that $q^{1/2}$ is not in $L(a, \infty)$. Furthermore, we may show that $q'/q^{3/2} \to 0$ as $t \to \infty$ by using the same argument that was used to show that $r'/r^{4/3} \to 0$ as $t \to \infty$ in Theorem 3. These two facts tell us that $\delta \to \infty$ as $t \to \infty$ and hence $\psi_2 \to +\infty$ and $\psi_3 \to -\infty$ as $t \to \infty$. Therefore, y_2 and y_3 are oscillatory on $[t_0, \infty)$ and hence on $[a, \infty)$.

Parts (i), (ii), and (iii) follow readily from (5) and (6).

An example of an equation covered by this theorem but no other known theorem is

$$y''' + t^3 y' - t \sin t y = 0.$$

This equation is oscillatory and its solutions satisfy (i), (ii), and (iii).

References

1. J. H. Barrett, Oscillation theory of ordinary linear differential equations, *Advances in Math.* **3** (1969), 415–509.
2. M. Hanan, Oscillation criteria for third-order linear differential equations, *Pacific J. Math.* **11** (1961), 919–944.
3. D. B. Hinton, Asymptotic behavior of solutions of $(ry^{(m)})^{(k)} \pm qy = 0$, *J. Differential Equations*, **4** (1968), 590–596.
4. A. C. Lazer, The behavior of solutions of the differential equation $y''' + py' + qy = 0$, *Pacific J. Math.* **17** (1966), 435–466.
5. G. W. Pfeiffer, Asymptotic solutions of $y''' + qy' + ry = 0$, *J. Differential Equations* **11** (1972), 145–155.
6. C. A. Swanson, "Comparison and Oscillation Theory." Academic Press, New York, 1968.